Form Uniqueness for Graphs with Weakly Spherically Symmetric Ends

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Setting

Let X be a countable vertex set. Any function $m\colon X \to (0,\infty)$ defines a measure of full support on X called the *vertex measure*. A *graph* over (X,m) is a pair (b,c), for killing term $c\colon X \to [0,\infty)$ and edge weight $b\colon X\times X \to [0,\infty)$, such that b is symmetric, has no loops, and $\sum_{y\in X} b(x,y) < \infty$ for all $x \in X$.

For a subset $O \subseteq X$ and $r \in \mathbb{N}$, define the edge boundary of the ball radius r

$$\partial B(r) = \sum_{x\in S_r}\sum_{y\in S_{r+1}} b(x,y)$$

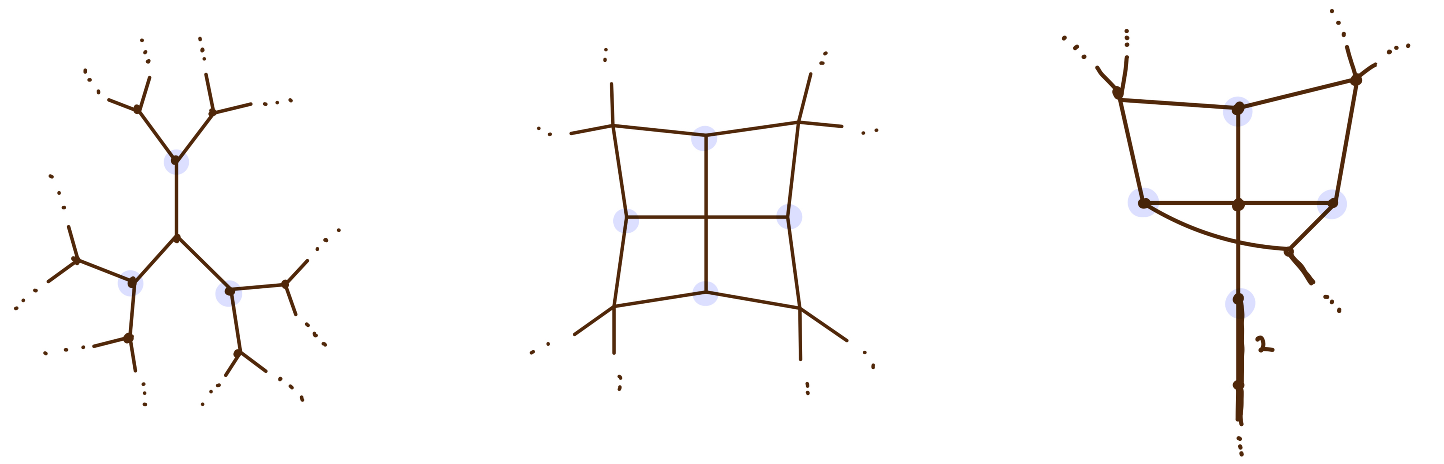
Weak spherical symmetry

We say that a function $f \in C(X)$ is *spherically symmetric* with respect to $O \subseteq X$ if for all $r\in\mathbb{N}$, $f(x)=f(y)$ for all $x,y\in S_r$. For $x \in S_r$, we let

$$\kappa_\pm(x) = \frac{1}{m(x)}\sum_{y\in S_{r\pm1}} b(x,y)$$

denote the *outer* and *inner vertex degrees*, respectively. We also define $q(x)=c(x)/m(x)$.

We say that a locally finite connected graph (b,c) over (X,m) is *weakly spherically symmetric* if there exists a finite set $O\subseteq X$ such that $\kappa_\pm$ and q are spherically symmetric.



Energy form

Define the *energy form*

$$\mathcal{Q}(f,g) = \frac{1}{2}\sum_{x,y\in X} b(x,y)(f(x)-f(y))(g(x)-g(y)) + \sum_{x\in X} c(x)f(x)g(x)$$

and let $\mathcal{D} = \{f \in C(X) \mid \mathcal{Q}(f) < \infty\}$ denote the space of *functions of finite energy*.

Let $Q^{(D)}$ be the restriction of $\mathcal{Q}$ to $D(Q^{(D)}) = \overline{C_c(X)}^{\|\cdot\|_{\mathcal{Q}}}$ where $\|\varphi\|_{\mathcal{Q}}^2 = \|\varphi\|^2 + \mathcal{Q}(\varphi)$ is the form norm. Let $Q^{(N)}$ denote the restriction of $\mathcal{Q}$ to $D(Q^{(N)}) = \mathcal{D}\cap\ell^2(X,m)$. We say that a graph satisfies *form uniqueness* if $Q^{(D)} = Q^{(N)}$.

We say that $u \in \mathcal{F}$ is *α-harmonic* for $\alpha\in\mathbb{R}$ if $(\Delta+\alpha)u = 0$.

Technical lemma (3.2 in [3])

Failure of form uniqueness is equivalent to existence of a nontrivial $u \geq 0$ and $\alpha > 0$ with $u \in \mathcal{D}\cap\ell^2(X,m)$ such that $(\Delta+\alpha)u=0$.

Operators

We let $\mathcal{F} = \{f\in C(X) \mid \sum_{y\in X} b(x,y)|f(y)| < \infty\}$, and for $f \in \mathcal{F}$ and $x\in X$, we let

$$\Delta f(x) = \frac{1}{m(x)}\sum_{y\in X} b(x,y)(f(x)-f(y)) + \frac{c(x)}{m(x)}f(x)$$

denote the *formal Laplacian*.

For $\varphi,\psi \in C_c(X)$ we get the Green's formula $\mathcal{Q}(\varphi,\psi) = \langle\Delta\varphi,\psi\rangle = \langle\varphi,\Delta\psi\rangle$.

We define the *formal averaging operator* as

$$\mathcal{A}f(x) = \frac{1}{m(S_r)}\sum_{x\in S_r} f(x)m(x)$$

for $x\in S_r$ and denote the restriction of $\mathcal{A}$ to $\ell^2(X,m)$ by A.

A locally finite graph (b,c) over (X,m) is weakly spherically symmetric if and only if $\Delta\mathcal{A} = \mathcal{A}\Delta$ on $C_c(X)$ (Lemma 9.8 in [3]).

Characterization via graph structure

Let (b,c) be a weakly spherically symmetric graph over (X,m). Then failure of form uniqueness is equivalent to $(c+m)(X)<\infty$ and $\sum_{r=0}^{\infty}\frac{1}{\partial B(r)} < \infty$.

Lemma: Averaging and energy

Let (b,c) be a weakly spherically symmetric graph over (X,m). If $v\in C(X)$, then $\mathcal{Q}(\mathcal{A}v) \leq \mathcal{Q}(v)$. In particular, if $v \in \mathcal{D}$, then $\mathcal{A}v\in\mathcal{D}$.

Lemma: Spherically symmetric energy

Let (b,c) be a weakly spherically symmetric graph over (X,m). Let u be spherically symmetric, non-zero and satisfying $(\Delta+\alpha)u=0$ for $\alpha>0$. Then $u\in\mathcal{D}$ if and only if

$$c(X)<\infty \quad \text{and} \quad \sum_{r=0}^{\infty}\frac{(m(B_r))^2}{\partial B(r)} < \infty.$$

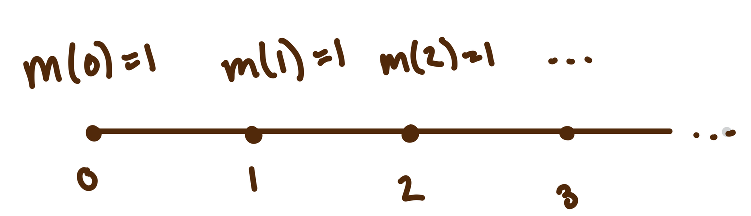


Figure 1. $\mathbb{N}$ with $m(\mathbb{N})=\infty$ has form uniqueness

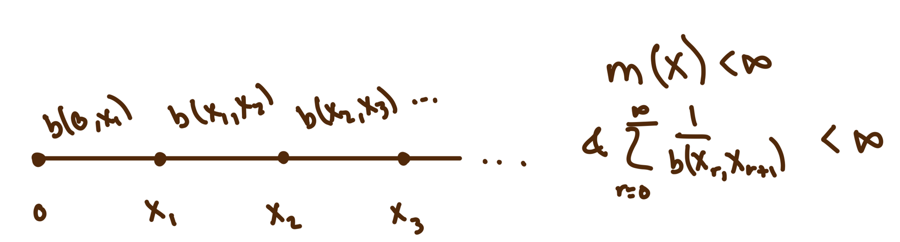


Figure 2. $\mathbb{N}$ without form uniqueness

Capacity of Cauchy boundary

A metric $d_\sigma : X\times X \to [0,\infty)$ is a *path metric* if $\sigma : X\times X\to[0,\infty)$ satisfies $\sigma(x,y)>0$ if and only if $b(x,y)>0$, and $d_\sigma(x,y)$ is the infimum of the σ-lengths of the paths connecting x and y. In addition, d_σ is called *strongly intrinsic* if

$$\sum_{y\in X} b(x,y)\sigma^2(x,y) \leq m(x)$$

for all $x\in X$.

For a set $K\subseteq X$, we let

$$\text{cap}(K) = \inf\{\|u\|_{\mathcal{Q}} \mid u\in D(Q^{(N)}) \text{ with } u\geq 1 \text{ on } K\}.$$

For a set $K\subseteq \overline{X}^\sigma$, we let

$$\text{cap}(K) = \inf\{\text{cap}(U) \mid K\subseteq U \text{ with } U\subseteq\overline{X}^\sigma \text{ open}\}.$$

Let $\partial_\sigma X = \overline{X}^\sigma\setminus X$ denote the *Cauchy boundary* of the graph.

Characterization via capacity

Let (b,c) be a weakly spherically symmetric graph over (X,m). Let d_σ be a strongly intrinsic path metric. Then failure of form uniqueness is equivalent to $0<\text{cap}(\partial_\sigma(X))<\infty$.

Weakly spherically symmetric ends

A graph (b,c) over (X,m) is a *a graph with weakly symmetric ends (for form uniqueness)* if $X_1\subseteq X$ can be chosen so that for $X_2 = X\setminus X_1$,

(1) $Q_1^{(D)} = Q_1^{(N)}$.
(2) Deg_∂ is bounded.
(3) (b_2,c_2) over (X_2,m_2) is a disjoint union of weakly spherically symmetric graphs.

Stability

Let (b,c) over (X,m) be a graph with weakly spherically symmetric ends for form uniqueness. Then failure of form uniqueness on X is equivalent to failure of form uniqueness on one weakly spherically symmetric subgraph in X_2.

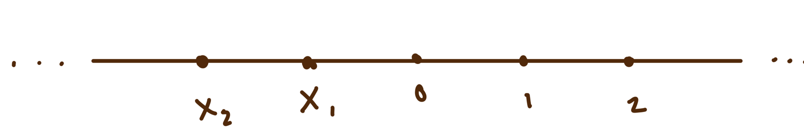


Figure 3. Gluing the two copies of $\mathbb{N}$ described earlier is not form unique.

References

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